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A PROCEDURE FOR CALCULATING VORTICITY BOUNDARY CONDITIONS IN THE STREAM-FUNCTION–VORTICITY METHOD

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SUMMARY

A new superconvergent projection formula for determining vorticity boundary data in the stream-function–vorticity method is constructed.

INTRODUCTION

The stream-function–vorticity (ψ, ω) formulation is a standard approach for numerical treatment of 2D viscous flows. In this procedure the problem reduces to solution of a coupled pair of partial differential equations – the vorticity transport equation and stream-function equation. These equations can be discretized and iteratively decoupled, and then solved for iterates approximating ψ and ω. A well-known difficulty in this algorithm is the problem of specifying vorticity boundary data as essential data for the vorticity transport equation. It is standard practice on rectangular finite-difference grids to use one-sided finite differences of the stream-function iterate to compute an approximation to the velocity and thereby the boundary vorticity.[1] A similar procedure can be used in finite-element methods but does not fit naturally in this framework. Here we present an alternative approach based on superconvergent flux ideas that applies to both straight and curved boundaries and can be used for either finite-element or finite-difference computations.

FORMULATION

Recall that in the stream-function–vorticity method the stream function ψ satisfies the Poisson equation

$$-\Delta\psi = \omega \text{ in } \Omega \tag{1}$$

Here ω is the vorticity determined from the vorticity transport equation (e.g. for steady Stokes flow $\Delta\omega = f$). The stream-function–vorticity equations are frequently iteratively decoupled in the numerical solution scheme. The objective here is to construct a proceudre that exploits superconvergence ideas to develop a post-processing formula from (1) for approximating the vorticity on the boundary. This can then be used as data for the vorticity transport equation.

We introduce the familiar Green–Gauss formula for the Laplacian operator

$$\int_\Omega (-\Delta u) v \, \mathrm{d}x \, \mathrm{d}y = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}y - \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, \mathrm{d}s \tag{2}$$

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where u, v are arbitrary admissible functions. In particular, let us select $u = \psi$ satisfying (1) so that

$$\int_{\Omega} \omega v \, dx \, dy = \int_{\Omega} \nabla\psi \cdot \nabla v \, dx \, dy - \int_{\partial\Omega} v \frac{\partial \psi}{\partial n} \, ds \tag{3}$$

Now (3) is an identity satisfied by the solution (ψ, ω) for arbitrary admissible v. In previous superconvergence studies, a similar construction has been developed and the approximate solution introduced to obtain superconvergent boundary flux approximations (i.e. for $\partial\psi/\partial n$) (e.g. see Wheeler[2], Carey[3]). In the present case we instead use the known boundary data $\partial\psi/\partial n = u_s$, where s is the tangential direction. Then for known ψ and $\partial\psi/\partial n$ in (3) we have a projection formula for vorticity in Ω. Now set the approximate solution ψ_h for ψ on the discretized domain Ω_h with $v = \phi_i$, the piecewise-polynomial Lagrange basis function associated with node i on the boundary, to get the approximate projection (for $\omega^* \approx \omega$)

$$\int_{\Omega_h} \omega^* \phi_i \, dx \, dy = \int_{\Omega_h} \nabla\psi_h \cdot \nabla\phi_i \, dx \, dy - \int_{\partial\Omega_h} u_s \phi_i \, ds \tag{4}$$

Now as i traverses the boundary nodes the integral on the left involves only the strip of elements adjacent to the boundary. Furthermore, since $\phi_i(x_j, y_j) = \delta_{ij}$, using a Lobatto (node point) quadrature on the left simplifies the expression to yield an explicit superconvergent extraction formula approximating the vorticity at boundary node i within quadrature accuracy as

$$Q_i \omega_i^* = \int_{\Omega_h} \nabla\psi_h \cdot \nabla\phi_i \, dx \, dy - \int_{\partial\Omega_h} u_s \phi_i \, ds \tag{5}$$

where Q_i corresponds to the accumulated quadrature weight at node i from the adjacent elements.

Remarks

1. For a rectilinear boundary and bilinear elements the extraction formula (5) is equivalent to a one-sided second-order difference approximation.[4] For higher-degree elements, more general boundary shapes and irregular grids, (5) is still applicable. The scheme has been applied in finite-element calculations for viscous flow applications with straight and curved boundary geometries as well as moving surfaces (Murray[5]).
2. The scheme (5) can be applied with finite-difference methods by formally introducing the nodal interpolant of the finite-difference solution as ψ_h.

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