

## Conservation Principles and Variational Problems

By

G. F. Carey, Austin, Texas, and H. T. Dinh, Berkeley, California

(Received March 1, 1985)

### Summary

We show formally that integral conservation equations can be related directly to corresponding integral variational statements, without the usual appeal to the classical differential form and the associated smoothness assumptions.

### Introduction

Integral relations describing conservation laws such as conservation of mass, momentum and energy are fundamental to the formulation of boundary-value problems and evolution problems in mechanics and elsewhere in the sciences. The fact that these relations hold over any arbitrary volume in the problem domain can then be used with appropriate smoothness assumptions to deduce differential equations such as the continuity equation and equations of motion that hold pointwise. Inclusion of an appropriate constitutive equation yields the governing equations for the classical mathematical statement of the problem, usually as a partial differential equation or system of equations. Analytic solutions can be developed for linear problems on simple domains with smooth data and for some other exceptional cases. More generally, approximate solutions to the differential equations based on finite difference methods or expansion techniques can be sought.

A variational formulation of a problem may be constructed from the preceding differential formulation by “pairing” the governing differential equations with a test function, in an integral statement, again under appropriate smoothness conditions on the admissible functions (Showalter [7]). This so called “weighted-residual statement” (e.g. see Finlayson [3]) corresponds, in the case of the variational theory of elasticity, to the well-known principle of virtual work.

Integration by parts in the weighted residual statement produces a (weak) variational statement of the problem. If a classical variational problem exists, then it is equivalent to the weak variational statement so obtained. When a classical solution exists for the differential equations, it follows that this is also a

solution of the variational problem. However, since the variational problem is an integral statement involving lower-order derivatives, the reverse is not true unless the solution to the variational problem is sufficiently smooth. Such weak variational statements form the foundation of Galerkin finite element methods for determining approximate solutions to boundary-value problems (e.g. see Carey and Oden [2], Oden and Carey [4]).

From the above remarks, we see that the variational methods are usually constructed from the governing differential equations which presume greater smoothness. These, in turn, are related to the underlying integral conservation relations. Our purpose here is to establish a direct formal relationship between the conservation relation and the variational problem without explicitly going through the usual intervening step involving the classical partial differential equation.

### • Analysis

Both the conservation relation and the variational problem are integral statements, the essential difference being that the conservation relation applies for an arbitrary volume in the domain whereas the variational problem involves an arbitrary admissible test function.

We begin with the basic transport theorem for an integral which states (see e.g. Pearson [5], Shinbrot [6]), that for arbitrary volume  $V(t)$  and an Eulerian frame

$$\frac{d}{dt} \int_{V(t)} f dx = \int_{V(t)} \left( \frac{\partial f}{\partial t} + \nabla \cdot (uf) \right) dx \quad (1)$$

where  $d/dt$  is the material derivative,  $dx$  denotes the infinitesimal volume  $dx_1 dx_2 dx_3$ ,  $u = dx/dt$  is the velocity and  $f$  is the transport variable (e.g. species concentration).

The corresponding conservation law is a balance relation obtained by setting the rate of change of the integral as defined in (1) equal to prescribed data. For example, if we set  $f = \rho$ , the density of the fluid in  $V$ , then in the absence of sources or sinks of mass in  $V$ , mass conservation implies that the integral in (1) must be zero. Similarly, we can take the momentum  $\rho u_i$ ,  $i = 1, 2, 3$  for  $f$  in (1) and equate rate of change of momentum to applied forces to obtain the equations of motion for a fluid or solid.

For clarity we shall consider a fundamental, frequently encountered case, it being clear that the ideas carry over immediately to more general systems. We note that if an equilibrium (steady) problem is considered  $\partial f/\partial t = 0$  and setting  $q = uf$  the conservation equation simplifies to

$$\int_V \nabla \cdot q dx = 0 \quad \text{for all } V \quad (2)$$

where  $V$  is an arbitrary volume in the fixed flow domain  $\Omega$ . Now we seek to take the relation (2) for arbitrary  $V$  in  $\Omega$  and construct a variational statement involving in arbitrary test function defined on  $\Omega$ .

Let  $V_1, V_2, \dots, V_N$  be any disjoint measurable subsets covering  $\Omega$  and let  $\chi_i$  be the characteristic function corresponding to  $V_i$ ,  $i = 1, 2, \dots, N$  defined by

$$\chi_i(x) = \begin{cases} 1 & \text{if } x \in V_i \\ 0 & \text{if } x \notin V_i. \end{cases} \quad (3)$$

By the definition of  $\chi_i$  in (3),

$$\int_{\Omega} (\nabla \cdot \mathbf{q}) \chi_i dx = \int_{V_i} (\nabla \cdot \mathbf{q}) \chi_i dx = \int_{V_i} \nabla \cdot \mathbf{q} dx. \quad (4)$$

But (2) holds for arbitrary  $V$  and thus holds for  $V = V_i$ , whence the integral on the right in (4) is zero and we have

$$\int_{\Omega} (\nabla \cdot \mathbf{q}) \chi_i dx = 0. \quad (5)$$

Now, for any constants  $\alpha_i$ ,

$$\int_{\Omega} (\nabla \cdot \mathbf{q}) \left( \sum_{i=1}^N \alpha_i \chi_i \right) dx = \sum_{i=1}^N \alpha_i \int_{\Omega} (\nabla \cdot \mathbf{q}) \chi_i dx$$

so that, by (5),

$$\int_{\Omega} (\nabla \cdot \mathbf{q}) \left( \sum_{i=1}^N \alpha_i \chi_i \right) dx = 0. \quad (6)$$

Next take  $\psi \in C_0^\infty(\Omega)$ , the space of infinitely differentiable test functions with compact support in  $\Omega$  (see, e.g. Adams [1] or Showalter [7]). Since the step functions  $\sum_{i=1}^N \alpha_i \chi_i$  in (6) are dense in  $C_0^\infty(\Omega)$  we can choose  $\sum_{i=1}^N \alpha_i \chi_i$  so that, for arbitrary  $\varepsilon$ ,

$$\left\| \psi - \sum_{i=1}^N \alpha_i \chi_i \right\|_{\infty} \leq \varepsilon. \quad (7)$$

Hence,

$$\left| \int_{\Omega} (\nabla \cdot \mathbf{q}) \psi dx \right| = \left| \int_{\Omega} (\nabla \cdot \mathbf{q}) \left( \psi - \sum_{i=1}^N \alpha_i \chi_i \right) dx \right| \leq \varepsilon \int_{\Omega} |\nabla \cdot \mathbf{q}| dx. \quad (8)$$

Letting  $\varepsilon \rightarrow 0$  we find

$$\int_{\Omega} (\nabla \cdot \mathbf{q}) \psi dx = 0 \quad \text{for all } \psi \in C_0^\infty(\Omega) \quad (9)$$

which corresponds to the "pairing" in our weighted residual statement (and which implies that  $\nabla \cdot \mathbf{q} = 0$  in a distributional sense).

Finally, since  $C_0^\infty(\Omega)$  is dense in  $H_0^1(\Omega)$ , we can select  $\psi$  in  $C_0^\infty(\Omega)$  so that, for  $v \in H_0^1(\Omega)$ ,

$$\|v - \psi\|_1 \leq \varepsilon. \quad (10)$$

Then, using Schwarz's inequality

$$\begin{aligned} \left| \int_{\Omega} (\nabla \cdot \mathbf{q}) v \, dx \right| &= \left| \int_{\Omega} (\nabla \cdot \mathbf{q}) (v - \psi) \, dx \right| \\ &\leq \|\nabla \cdot \mathbf{q}\|_0 \|v - \psi\|_0 \leq \|\nabla \cdot \mathbf{q}\|_0 \|v - \psi\|_1 \leq \varepsilon \|\nabla \cdot \mathbf{q}\|_0. \end{aligned} \quad (11)$$

For  $\varepsilon \rightarrow 0$ , it follows that

$$\int_{\Omega} (\nabla \cdot \mathbf{q}) v \, dx = 0 \quad (12)$$

and using Green's identity we have

$$\int_{\Omega} \mathbf{q} \cdot \nabla v \, dx = 0 \quad \text{for all } v \in H_0^1(\Omega) \quad (13)$$

which constitutes the integral statement for the weak variational problem.

### Concluding Remarks

In the best known case  $\mathbf{q}$  is derivable from a potential  $\phi$  with  $\mathbf{q} = \nabla\phi$  and in (13) we have the variational problem: Find  $\phi \in H^1(\Omega)$  satisfying the essential boundary conditions  $\phi = g$  on a sufficiently smooth boundary  $\partial\Omega$  and such that

$$\int_{\Omega} \nabla\phi \cdot \nabla v \, dx = 0 \quad \text{for all } v \in H_0^1(\Omega). \quad (14)$$

More generally, in for example compressible gas flows and nonlinear porous flows, we have a constitutive relation of the form  $\mathbf{q} = k\nabla\phi$  where  $k = k(|\nabla\phi|)$ . The variational statement is obtained by replacing  $\nabla\phi$  by  $k\nabla\phi$  in (14) and using appropriate Sobolev spaces. The foregoing analysis can evidently be applied to other conservation relations and easily extended to the unsteady case.

### Acknowledgments

This research has been supported in part by the Department of Energy and the Center for Oil and Gas Recovery Research at the University of Texas.

## References

- [1] Adams, R. A.: Sobolev spaces. New York: Academic Press 1975.
- [2] Carey, G. F., Oden, J. T.: Finite elements — a second course. New York: Prentice Hall 1983.
- [3] Finlayson, B. A.: The method of weighted residuals and variational principles. New York: Academic Press 1972.
- [4] Oden, J. T., Carey, G. F.: Finite elements — mathematical aspects. New York: Prentice Hall 1984.
- [5] Pearson, C. E.: Theoretical elasticity. Cambridge, Massachusetts: Harvard University Press 1959.
- [6] Shinbrot, M. S.: Lectures on fluid mechanics. London: Gordon and Breach 1973.
- [7] Showalter, R. E.: Hilbert space methods for partial differential equations. Pitman 1977.

*G. F. Carey*  
*Engineering Mechanics*  
*University of Texas*  
*Austin*  
*Texas*  
*U.S.A.*

*H. T. Dinh*  
*Mathematics Department*  
*University of California*  
*Berkeley*  
*California*  
*U.S.A.*

