EXPONENTIAL UPWINDING AND INTEGRATING FACTORS FOR SYMMETRIZATION

G.F. CAREY
University of Texas at Austin, U.S.A.

SUMMARY

Exponential upwinding in the Petrov–Galerkin method for convection diffusion problems is related to the use of integrating factors for transforming the problem to a (symmetric) self-adjoint form. The technique yields non-oscillatory approximations on coarse meshes.

INTRODUCTION

Both convection and diffusion enter as the important distinct effects governing transport processes. The introduction of approximations in numerical methods influences the relative importance of these two effects and can produce approximate solutions that are too dissipative or which exhibit numerical oscillations.

Finite difference and finite element methods have been applied extensively to these problems. In the finite difference method, the need to control oscillations has led to the use of backward difference operators for the convective term (see for example, Reference 1). Similarly motivated, Petrov–Galerkin methods have been employed with upwind biased test functions to produce finite element methods that are not oscillatory. A variety of upwind basis functions have been suggested and applied in pilot studies to the model convection–diffusion equation. Several examples may be found in the A.S.M.E. monograph edited by Hughes.[2] Quadratic upwind test functions have been used relatively frequently since they are simple to construct. More recently, Szymcak and Babuška[3] have obtained estimates which imply that exponential upwind test functions may be preferable for the model problem if the forcing data and coefficients are not well behaved. In the present study, we examine the use of integrating factors for symmetrizing the model convection–diffusion problem and relate this to exponential upwinding.

DISCUSSION

The model convection–diffusion equation on $0 < x < 1$, $t > 1$

$$\frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial x} - \beta^2\frac{\partial^2 u}{\partial x^2} = f \tag{1}$$

and the steady-state form

$$\alpha\frac{\partial u}{\partial x} - \beta^2\frac{\partial^2 u}{\partial x^2} = f \tag{2}$$

where $\alpha, \beta^2 > 0$ are constants, have been frequently studied as prototype problems for developing effective numerical techniques for transport processes. Initial conditions for (1) and boundary conditions for (1) and (2) complete the mathematical statements of the problems.

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A Petrov–Galerkin finite element approximation is obtained by introducing different trial and test approximation spaces in the variational statements corresponding to (1) and (2).[4,5] For simplicity, let us consider the case of Dirichlet data. The approximate problem corresponding to (1) is: for any $t > 0$ find $u_h \in H^h$ satisfying the initial and boundary conditions and such that

$$\int_0^1 \left(\frac{\partial u_h}{\partial t} v_h + \alpha \frac{\partial u_h}{\partial x} v_h + \beta^2 \frac{\partial u_h}{\partial x} \frac{\partial v_h}{\partial x} \right) \mathrm{d}x = \int_0^1 f v_h \mathrm{d}x \text{ for all } v_h \in \hat{H}^h \tag{3}$$

and for (2): find $u_h \in H^h$ satisfying the boundary conditions and such that

$$\int_0^1 \left(\alpha \frac{\partial u_h}{\partial x} v_h + \beta^2 \frac{\partial u_h}{\partial x} \frac{\partial v_h}{\partial x} \right) \mathrm{d}x = \int_0^1 f v_h \mathrm{d}x \text{ for all } v_h \in \hat{H}^h \tag{4}$$

The most common case is the piecewise linear basis for H^h and the quadratically unwinded basis for $\hat{H}^h$ with scalar upwinding parameter w. Griffiths and Lorenz[4] have shown that the upwind parameter can be chosen to yield a best 'quasi-optional' estimate of the form

$$|u - u_h|_1 \leqq C \inf_{w_h \in H^h} |u - w_h|_1 \tag{5}$$

where u is the exact solution, u_h is the approximate solution and $|\cdot|_1$ is the L^2-norm of the derivative (the H^1 semi-norm). However, for general data f, the constant C in this estimate is not bounded uniformly in $\epsilon = \beta^2/\alpha$ and h. In fact, if $\epsilon \ll h$ then $C \cong 1/h$ as $h \to 0$. Szymcak and Babuška have subsequently shown that the use of an exponential test basis in (4) will yield the estimate (5) with C independent of ϵ and h, but in a non-standard norm.

The matrices arising from (3) and (4) for the standard Galerkin method include a skew-symmetric matrix from the convection term and a symmetric matrix from the self-adjoint diffusion term. If $\alpha \gg \beta^2$ and h is not sufficiently small, the resulting discrete system is not diagonally dominant and the solution may be oscillatory.

Since the spatial operator is linear, the problem can easily be transformed to a 'conservative' form that is self-adjoint by introducing an integrating factor. For the model problem, the integrating factor is $\exp(-\alpha x/\beta^2)$, so that (1) and (2) become

$$\mathrm{e}^{\frac{-\alpha x}{\beta^2}} \frac{\partial u}{\partial t} + \left(-\beta^2 \mathrm{e}^{\frac{-\alpha x}{\beta^2}} u_x \right)_x = \mathrm{e}^{\frac{-\alpha x}{\beta^2}} f \tag{6}$$

and

$$\left(-\beta^2 \mathrm{e}^{\frac{-\alpha x}{\beta^2}} u_x \right)_x = \mathrm{e}^{\frac{-\alpha x}{\beta^2}} f \tag{7}$$

The corresponding Galerkin finite element approximations to (6) and (7) are

$$\int_0^1 \left(\mathrm{e}^{\frac{-\alpha x}{\beta^2}} \frac{\partial u_h}{\partial t} v_h + \beta^2 \mathrm{e}^{\frac{-\alpha x}{\beta^2}} \frac{\partial u_h}{\partial x} \frac{\partial v_h}{\partial x} \right) \mathrm{d}x = \int_0^1 \mathrm{e}^{\frac{-\alpha x}{\beta^2}} f v_h \mathrm{d}x \tag{8}$$

and

$$\int_0^1 \beta^2 \mathrm{e}^{\frac{-\alpha x}{\beta^2}} \frac{\partial u_h}{\partial x} \frac{\partial v_h}{\partial x} \mathrm{d}x = \int_0^1 \mathrm{e}^{\frac{-\alpha x}{\beta^2}} f v_h \mathrm{d}x \tag{9}$$

where the same basis functions are used for trial and test functions.

To relate the conservative form to the upwind Petrov–Galerkin method, set $\epsilon = \beta^2/\alpha$ and note that after integration by parts and simplification,

$$\int_0^1 \beta^2 \mathrm{e}^{-\frac{x}{\epsilon}} u' v' \mathrm{d}x = \int_0^1 (\alpha u' - \beta^2 u'') \mathrm{e}^{-\frac{x}{\epsilon}} v \mathrm{d}x \tag{10}$$

Let us consider the steady-state formulation (9).† Writing $\tilde{v} = e^{-\frac{x}{\epsilon}} v$ and integrating by parts as in (10), this implies that (9) is equivalent to

$$\int_0^1 (\alpha u_h' \tilde{v}_h + \beta^2 u_h' \tilde{v}_h)\mathrm{d}x = \int_0^1 f\tilde{v}_h \mathrm{d}x \tag{11}$$

where $\tilde{v}_h \equiv e^{-\frac{x}{\epsilon}} v_h$ and v_h is the usual (e.g. piecewise-linear) test basis. Now, since v_h is an arbitrary member of the test space H^h,

$$v_h(x) = \sum_{j=1}^{N} b_j \phi_j(x)$$

where $\phi_j(x)$ are piecewise-Lagrange polynomials. Then

$$\tilde{v}_h(x) = \sum_{j=1}^{N} b_j e^{\frac{-x}{\epsilon}} \phi_j(x) = \sum_{j=1}^{N} b_j e^{\frac{-x_j}{\epsilon}} e^{\frac{-(x-x_j)}{\epsilon}} \phi_j(x)$$

$$= \sum_{j=1}^{N} C_j \tilde{\phi}_j(x) \, , \; \tilde{\phi}_j = e^{\frac{-(x-x_j)}{\epsilon}} \phi_j \tag{12}$$

where $\{\tilde{\phi}_j(x)\}$ are exponentially upwinded test functions centred at x_j and b_j are arbitrary. Since (11) must hold for all $\tilde{v}_h \in \tilde{H}_h$, it suffices that we consider the upwind basis $\{\tilde{\phi}_j\}$ for v_h in (11).

Hence, it follows that the conservative (integrating factor) formulation using the standard Galerkin method in (9) is equivalent to a Petrov–Galerkin scheme with exponential upwinding. The same result holds for the unsteady problem. This exponential test function differs slightly from that of Szymcak and Babuška, who instead solve a boundary-value problem for the adjoint operator to obtain the exponential test functions.

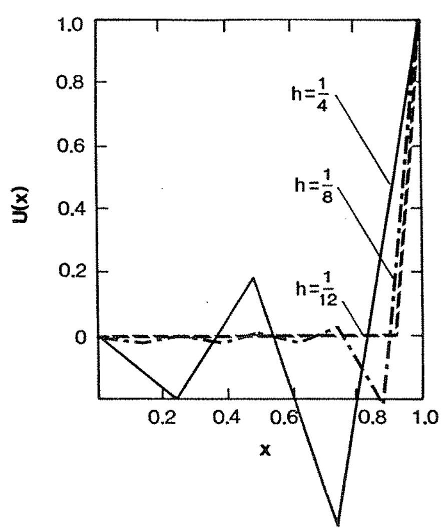

Figure 1. Oscillatory approximations for model problem and standard Galerkin method (9) with $\alpha = 1$, $\beta^2 = 1/24$

As an example, let us take the steady-state problem (2) with $\alpha = 1$, $\beta^2 = 1/24, f = 0, u(0) = 0, u(1) = 1$ and a piecewise linear basis. The standard Galerkin method (4) for the non-conservative form ($H^h = \hat{H}^h$) produces oscillatory approximations on coarse meshes ($h < 1/12$), as in Figure 1.

†The analysis for the unsteady problem (8) is essentially the same.

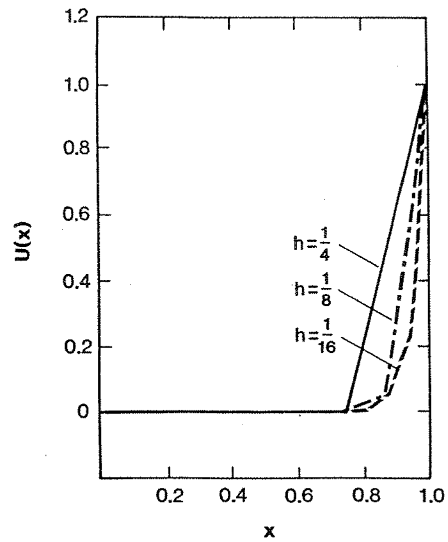

Figure 2. Non-oscillatory solutions to model problem with symmetrization using the integrating factor or equivalent exponential upwinding

The Galerkin method for the conservative form in (9) is not oscillatory, as seen in the results of Figure 2.

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REFERENCES

1. P. Roache, *Computational Fluid Dynamics,* Hermosa Press, 1982.
2. T.J.R. Hughes (Ed.), *Finite Element Methods for Convection-Dominated Flows,* A.S.M.E. Monograph AMD 34, 1979.
3. W.G. Szymcak and I. Babuška, 'An analysis of a finite element method for convection diffusion problems', *Tech. Notes BN-1001, BN-1002,* Inst. for Physical Science and Technology, Univ. of Maryland (March 1983).
4. D.F. Griffiths and J. Lorenz, 'An analysis of the Petrov–Galerkin finite element method', *Comp. Meth. Appl. Mech. Eng.,* **14**, 39–64 (1978).
5. J. Christie, D.F. Griffiths, A.R. Mitchell and O.C. Zienkiewicz, 'Finite element methods for second order equations with significant first derivative', *Int. j. numer. methods eng.,* **10**, 1389–1396 (1976).