SHORT COMMUNICATION

C^1 – collation and continuity requirements

G. F. Carey

Texas Institute for Computational Mechanics, Department of Aerospace Engineering,
The University of Texas at Austin, Austin, Texas 78712, USA

(Received October 1981)

This communication deals with an inherent difficulty related to continuity in C^1 finite element collocation methods. It is shown that the C^1 condition is too special. Consequently these methods, as usually developed, are too restrictive and not suited to general purpose finite element programs, even for two-point problems. In view of this elementary observation, it may be argued that this approach has received more attention than this form of the method really merits, and that other (weaker) collocation procedures are probably more worthy.

Key words: mathematical analysis

Introduction

The interest in finite element collocation methods is mainly due to the ease with which they can be formulated for certain relatively simple domains and to the fact that no numerical integration is required in determining the system. There are also well-known disadvantages as compared with Galerkin finite element methods. These primarily concern the additional complexity of the basis, loss of symmetry and the use of the methods for irregular meshes in dimensions greater than one.

Collocation methods are an interesting subject in their own right. Moreover, there may be specific classes of problems for which they are suited. For these reasons, there have been several studies of both practical and theoretical nature concerning these methods (see, for example, the references cited). Many of the studies have considered the use of C^1 collocation for two-point problems and for second-order elliptic problems on regular domains in two dimensions. In particular, the piecewise Hermite basis functions in one dimension and the tensor product Hermites in two dimensions are usually introduced to construct a C^1 basis. This construction provides a simple yet elegant means for generating the basis. The result is that the notion of C^1 continuity and the use of these bases is widely interpreted as an intrinsic part of these finite element collocation methods. However, a point that has not received adequate attention is that this construction produces a basis that limits the applicability of the method, even on simple domains such as an interval in the case of the two-point problem. This issue appears to have been largely overlooked, or perhaps simply not given due attention since some collocation programs purporting to be of a general-purpose nature have been developed and do not appear to address this issue. The objective of this note is to discuss the origin and nature of the difficulty and indicate briefly some alternative collocation techniques.

Discussion

The ideas can be easily described by considering the second-order ordinary differential equation:

$$-(au')' + bu = f \quad \text{in} \quad 0 < x < 1 \tag{1}$$

with homogeneous boundary conditions:

$$u(0) = u(1) = 0 \tag{2}$$

The coefficient a is assumed to be continuous with $a(x) \geqslant a_0 = \text{constant} > 0$ and b is assumed continuous at all points $x \in [0, 1]$.

As is well known, in a Galerkin finite element formulation of this problem only first derivatives appear in the integral expressions, so the admissible functions are required to have square-integrable first derivatives. A piecewise polynomial basis that is only continuous (C^0) at the interface nodes between elements is usually employed for the given problem.

Similarly, in a finite element collocation scheme for this example, we can use a piecewise-polynomial basis, but the global continuity requirements may differ depending on the collocation method. In collocation methods an admissible approximation u_h is introduced directly into the differential equation to define a residual and the residual is made zero at discrete collocation points x_c:

$$R = -(au_h')' + bu_h - f = 0 \quad \text{at} \quad x_c \tag{3}$$

Since the residual R is to be evaluated at a collocation point, this implies that the admissible functions should be locally C^2 since R involves second derivatives of u_h. This suggests that we collocate in the element interiors where the functions are smooth, but can relax the interelement continuity. If we relax the continuity between elements to be only C^1, then C^1 Hermite-type bases may be used and this will yield a C^1-collocation method. For example, if Hermite cubics are used on an element, the global approximation is C^1; u_j

0307-904X/82/010070-02/$03.00

and u_j' are the degrees-of-freedom at each node j, and the discrete system is obtained from the boundary conditions and collocating at any two distinct points in the interior of each element. Such an approach is quite valid for the given problem.

While it may appear that global C^1 continuity is a consequence of the collocation procedure, this is not really the case. C^1 continuity is more properly attributed to the actual physics of the problem and is appropriate here because the coefficient a is continuous on (0, 1)*. Equations such as (1) are derived from a conservation law and constitutive equation, and describe equilibrium in a physical process. The quantity $\sigma = au'$ is the flux, and, in the absence of point sources or sinks, it is the flux σ not u' that should be continuous across any interface x in that domain. That is:

$$[\![\sigma]\!] = [\![au']\!] = 0 \tag{4}$$

at any point x in (0, 1), where $[\![\cdot]\!]$ denotes the jump. If a is continuous, then equation (4) implies that $[\![u']\!] = 0$ so that u must be globally C^1, and a C^1 finite element basis will satisfy the requirement. Many problems, however, involve dissimilar materials and the coefficient a is no longer continuous. Then (4) must be enforced by other means, and C^1 continuity of the basis is quite inappropriate.

Now this simple point at first appears to be a minor issue. In fact, a similar difficulty arises in the Galerkin finite element method. However, there the consequences are less damaging. In the variational formulation of the boundary-value problem, the condition (4) on the flux is produced as a natural boundary condition by the integration-by-parts procedure. This implies only that we must restrict the mesh such that element end nodes coincide with points where the coefficient a is discontinuous. The Galerkin finite element method will then satisfy flux conservation at this point and, incidentally, C^1 continuity at other interface nodes where a is continuous (albeit approximately). A moment's inspection of a Galerkin finite element solution using, for example, a piecewise-linear basis reveals that au_h' is certainly not usually continuous at the nodes. The variational principle attempts to satisfy this condition as well as possible though, and automatically; thus, no special programming is involved. No such resolution exists for the collocation method with C^1 Hermite-type bases. Either special elements must be devised to be used at such points when a is not continuous, or completely different bases and special techniques for handling the flux continuity (4) must be used. One such approach is to use a C^0-Lagrange type basis as in the Galerkin finite element method. The local element polynomial should be of sufficient degree to permit collocation as before (at least cubic Lagrange). At the interface node i between elements e and $e + 1$, the flux condition (4) implies:

$$\lim_{x \to x_i^-} [au_e'(x)] = \lim_{x \to x_i^+} [au_{e+1}'(x)] \tag{9}$$

which yields the additional equations needed to complete the finite element collocation system. Other schemes exist for enforcing flux continuity in conjunction with interior collocation. C^0-collocation–Galerkin methods[11,12] are a particular example.

Concluding remarks

The global continuity of C^1-collocation finite element methods is, in a sense, too special and restricts the utility of collocation methods. In particular, they cannot be applied to problems in which the flux coefficient has a simple discontinuity. Since this situation arises frequently in practical applications where, for example, dissimilar materials are in contact, this is a severe limitation on the method. In view of this, other bases and alternative procedures for treating the flux continuity are preferable.

Acknowledgements

This research has been supported in part by the Office of Naval Research Grant ONR N00014-79-C-0550.

References

1 Ascher, U. *et al.* In 'Codes for boundary-value problems in ordinary differential equations', Lecture Notes in Computer Science, No. 76, Springer-Verlag, Berlin, 1979
2 Carey, G. F. and Finlayson, B. A. *J. Chem. Engng. Sci.* 1975, **30**, 569
3 de Boor, C. and Swartz, B. *SIAM J. Numer. Anal.* 1973, **10**, 482
4 de Boor, C. *SIAM J. Numer. Anal.* 1977, p. 441
5 Diaz, J. C. *SIAM J. Numer. Anal.* 1977, **14** (5), 844
6 Douglas Jr., J. and Dupont, T. *Math. Comp.* 1973, **27**, 17
7 Hayes, L. J. *Comp. Math. Applic.* 1980, **6**, 45
8 Houstis, E. N. *et al. Rep. CSD TR 121*, Purdue University, 1974
9 Madsen, N. K. and Sincovec, R. F. In 'Numerical methods for differential systems', Academic Press, New York, pp. 229–242, 1976
10 Madsen, N. K. and Sincovec, R. F. *Trans. Math. Software, ACM* 1978
11 Percell, P. and Wheeler, M. F. *SIAM J. Numer. Anal.* 1981 (to appear)
12 Pinder, G. F. *et al.* In *Proc. 2nd Int. Conf. Finite Elements in Water Resources*, C. A. Brebbia *et al.* (eds), Pentech Press, Plymouth, UK, pp. 1171–1185, 1978
13 Prenter, P. M. and Russell, R. D. *SIAM J. Numer. Anal.* 1976, **13**, 923
14 Sincovec, R. F. *J. Soc. Petrol. Eng.* 1977, **00**, 345
15 Wheeler, M. F. *SIAM J. Numer. Analysis* 1977, **14** (1), 71.

---

* It may also be deduced from the (weak) variational statement of the problem and the implied integrability conditions on the admissible functions.